Bab 2. Teknik Integrasi

2.1 Integral Parsial

berawal dari aturan Perkalian

$$\frac{d}{dx}[f(x).g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

atau apabila dalam u dan v

$$\frac{d}{dx}[u.v] = u'v + uv'$$

Dengan mengintegralkan kedua ruas, diperoleh :

$$\int \frac{d}{dx}[u.v]\,dx = \int u'v\,dx + \int uv'\,dx$$

$$u.v + C = \int u'v\,dx + \int uv'\,dx$$

$$\int uv'\,dx = u.v - \int u.v'\,dx$$

Sehingga, diperoleh rumus umum Integral Parsial adalah sebagai berikut.

$$\boxed{\int u\,dv = u.v - \int v.du}$$

▶ Konsep ILATE

ILATE merupakan prioritas fungsi yang dapat dimisalkan sebagai u

Invers	: $\tan^{-1}x$, $\sin^{-1}x$, ...	Semakin ke atas, semakin dapat diprioritaskan utk dimisalkan u.
Logaritma	: $\ln x$, $\ln x^2$, ...	
Aljabar	: x, x^2, $\frac{1}{x}$	
Trigonometri	: $\sin x$, $\cos x$	
Eksponensial	: e^x, ...	

Contoh soal :

• $\int t\tan^{-1}t\,dt = \dots$

① misalkan u

$u = \tan^{-1}t \longrightarrow du = \frac{1}{1+t^2}\,dt$

$dv = t\,dt \longrightarrow v = \frac{1}{2}t^2$

② Substitusi ke rumus :

$$\int t\tan^{-1}t\,dt = \frac{t^2.\tan^{-1}t}{2} - \int \frac{1}{2}t^2.\frac{1}{1+t^2}\,dt$$

※ $\frac{1}{2}\int \frac{t^2}{1+t^2}\,dt$

$$\frac{t^2}{1+t^2} = \frac{(t^2+1)-1}{t^2+1} = \frac{t^2+1}{t^2+1} - \frac{1}{t^2+1} = 1 - \frac{1}{t^2+1}$$

$$\int \frac{t^2}{1+t^2}\,dt = \int dt - \int \frac{1}{t^2+1}\,dt = t - \tan^{-1}t + C$$

• Substitusi ke soal :

$$\int t\tan^{-1}t\,dt = \frac{t^2\tan^{-1}t}{2} - \frac{1}{2}\left(t - \tan^{-1}t\right) + C = \frac{t^2\tan^{-1}t - t + \tan^{-1}t}{2} + C$$

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• $\int x^2e^{2x}\,dx$

misal : $u = x^2 \longrightarrow du = 2x\,dx$

$dv = e^{2x}dx \longrightarrow v = \frac{1}{2}e^{2x}$

$$\int x^2e^{2x}dx = \frac{x^2}{2}e^{2x} - \int \frac{1}{2}e^{2x}.\,2x\,dx \quad \dots\dots \text{(a)}$$

misalkan lagi untuk $\int \frac{1}{2}e^{2x}.\,2x\,dx$

misal : $u = x \longrightarrow du = dx$

$dv = e^{2x}dx \longrightarrow v = \frac{1}{2}e^{2x}$

$$\int e^{2x}.\,2x\,dx = \frac{xe^{2x}}{2} - \int \frac{1}{2}e^{2x}dx = \frac{xe^{2x}}{2} - \frac{1}{2}.\frac{1}{2}e^{2x} + C = \frac{1}{2}e^{2x}\left[x - \frac{1}{2}\right] + C \quad \dots \text{(b)}$$

Substitusi (b) ke (a)

$$\int x^2e^{2x}dx = \frac{x^2}{2}e^{2x} - \frac{1}{2}e^{2x}\left[x - \frac{1}{2}\right] + C$$

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$$\int \ln(t^2+1)\,dt$$

misal : $u = \ln(t^2+1) \longrightarrow du = \frac{1}{t^2+1} \cdot 2t\,dt$

$dv = dt \longrightarrow v = t$

$$\int \ln(t^2+1)\,dt = t.\ln(t^2+1) - \int \frac{2t^2}{t^2+1}\,dt \quad \text{... (a)}$$

※ penyelesaian $\int \frac{2t^2}{t^2+1}dt$

$$\frac{t^2}{t^2+1} = \frac{(t^2+1)-1}{t^2+1} = 1 - \frac{1}{t^2+1}$$

$$2\int \frac{t^2}{t^2+1}dt = 2\left[\int dt - \int \frac{1}{t^2+1}\,dt\right]$$

$$= 2\left[t - \tan^{-1}t\right] + C \quad \text{... (b)}$$

Substitusi (b) ke (a)

$$\int \ln(t^2+1)\,dt = t\ln(t^2+1) - 2\left[t - \tan^{-1}t\right] + C$$

▶ Latihan Soal

a). $\int te^{4t}\,dt$ c). $\int \ln(3x+1)\,dx$

b). $\int t \sin 2t\,dt$ d). $\int e^x \cos x\,dx$

※ Pengintegralan Sinus dan Cosinus

contoh soal :

$$\int \sin^2 3t \cos^3 3t\,dt =$$

karena n (pangkat cos) ganjil, maka :

$u = \sin 3t$

$du = 3\cos 3t\,dt$

$\frac{du}{3} = \cos 3t\,dt$

$$\int \sin^2 3t \cos^3 3t\,dt = \int u^2 . \cos^2 3t . \frac{du}{3}$$

$$= \frac{1}{3}\int u^2 \cos^2 3t$$

※ $\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$$= \frac{1}{3}\int u^2 (1-u^2)$$

$$= \frac{1}{3}\int (u^2 - u^4)\,du$$

$$= \frac{1}{3}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{3}\sin^3 3t - \frac{1}{5}\sin^5 3t\right] + C$$

▶ Latihan Soal

a). $\int \sin^3 x \cos^3 x\,dx$ b). $\int \sin^3 5x \cos^2 5x\,dx$

※ Pengintegralan Secan dan Tangen

$$\int \tan^5\theta \sec^4\theta\,d\theta$$

karena n (pangkat secan) genap, maka :

misal : $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

※ diketahui

$\sec^2 x = \tan^2 x + 1$

※ maka $\sec^2\theta = \tan^2\theta + 1$

$= u^2 + 1$

Substitusi ke soal :

$$\int \tan^5\theta \sec^4\theta\,d\theta = \int u^5 . \sec^2\theta . \sec^2\theta\,d\theta$$

$$= \int u^5 (u^2+1)\,.\,du$$

$$= \int (u^7 + u^5)\,du$$

$$= \frac{1}{8}u^8 + \frac{1}{6}u^6 + C$$

$$= \frac{1}{8}\tan^8\theta + \frac{1}{6}\tan^6\theta + C$$

Latihan Soal :

a). $\int \tan^4\theta \sec^4\theta\,d\theta$ b). $\int \tan^2\frac{\theta}{2} \sec^3\frac{\theta}{2}\,d\theta$

Soal Latihan 2.2

2). a. $\int \frac{dx}{x^2+4x-12} = \ldots$

I. Menentukan Bentuk Dekomposisi Pecahan

$$x^2+4x-12 = (x+6)(x-2)$$

Sehingga faktor-faktor linear :

$$\frac{1}{x^2+4x-12} = \frac{1}{(x+6)(x-2)}$$

$$\frac{1}{(x+6)(x-2)} = \frac{A}{(x+6)} + \frac{B}{(x-2)}$$

$$\frac{1}{(x+6)(x-2)} = \frac{A(x-2)+B(x+6)}{(x+6)(x-2)}$$

$$A(x-2) + B(x+6) = 1$$

$$Ax + Bx - 2A + 6B = 1$$

Diperoleh :

- $A + B = 0 \rightarrow A = -B$
- $-2A + 6B = 1$

$$8B = 1$$

$$B = \frac{1}{8} \rightarrow A = -\frac{1}{8}$$

$$\int \frac{dx}{x^2+4x-12} = \int\left(\frac{-1/8}{x+6} + \frac{1/8}{x-2}\right)dx$$

$$= -\frac{1}{8}\ln(x+6) + \frac{1}{8}\ln(x-2) + C$$

g). $\int \frac{3t^2+4}{t^2-4t+4}\,dt$

Pembagian Pecahan :

$$\begin{array}{r|l} & 3 \\ t^2-4t+4 & 3t^2+4 \\ & 3t^2-12t+12 \quad - \\ \hline & 12t-8 \end{array}$$

hasil pembagian :

$$\frac{3t^2+4}{t^2-4t+4} = 3 + \frac{12t-8}{t^2-4t+4}$$

$$\int 3 + \frac{12t-8}{t^2-4t+4}\,dt$$

$$* \int \frac{12t-8}{t^2-4t+4}\,dt$$

Menentukan Dekomposisi Pecahan

$t^2-4t+4 = (t-2)(t-2)$, sehingga :

$$\frac{12t-8}{(t-2)^2} = \frac{A}{t-2} + \frac{B}{(t-2)^2}$$

$$12t-8 = A(t-2) + B$$

$$12t-8 = At - 2A + B$$

Didapatkan :

$$A = 12$$

$$-2A + B = -8$$

$$-24 + B = -8$$

$$B = 16$$

$$\int \frac{12t-8}{t^2-4t+4}\,dt = \int \frac{12}{t-2} + \frac{16}{(t-2)^2}\,dt$$

$$= 12\ln|t-2| + \left(-\frac{16}{(t-2)}\right) + C$$

$$* \int \frac{16}{(t-2)^2}\,dt \rightarrow \int \frac{16}{u^2}\,du = 16\left[-1\,u^{-1}\right]$$

misal : $u = t-2$

$du = dt$

$$= -\frac{16}{u} + C$$

$$\int 3 + \frac{12t-8}{t^2-4t+4}\,dt = 3t + 12\ln|t-2| - \frac{16}{(t-2)} + C$$

Latihan Soal !

a). $\int \frac{dx}{x^2-6x+5}$ b). $\int \frac{3t^2}{t^2-3t+2}\,dt$

$$\int \frac{t^3}{t^2-3t+2}\,dt$$

Pembagian pecahan

$$\begin{array}{r|l} & t+3 \\ t^2-3t+2 & t^3 \\ & t^3-3t^2+2t \quad - \\ \hline & 3t^2-2t \\ & 3t^2-9t+6 \quad - \\ \hline & 7t-6 \end{array}$$

$$\int \frac{t^3}{t^2-3t+2}\,dt = \int (t+3) + \frac{7t-6}{t^2-3t+2}\,dt$$

* Dekomposisi Pecahan

$$\frac{7t-6}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}$$

$$A(t-1) + B(t-2) = 7t-6$$

$$At + Bt - A - 2B = 7t-6$$

$$(A+B)t - (A+2B) = 7t-6$$

$\Rightarrow A+B = 7$

$A = 7-B$

$\Rightarrow A+2B = 6$

$7-B+2B = 6$

$7+B = 6$

$B = -1$

$A = 7-B$

$= 7-(-1)$

$= 8$

$$\int \frac{7t-6}{t^2-3t+2}\,dt = \int \frac{8}{t-2}\,dt + \int -\frac{1}{t-1}\,dt$$

$$= 8\ln|t-2| + (-\ln|t-1|) + C$$

$$\int (t+3) + \frac{7t-6}{t^2-3t+2}\,dt = \frac{1}{2}t^2 + 3t + 8\ln|t-2| - \ln|t-1| + C$$

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$$\int \frac{t^2+1}{(t^2+2t+3)^2}\,dt$$

▶ .. Ubah Penyebut Jadi Kuadrat Sempurna

$$t^2+2t+3 = (t+1)^2+2$$

misal : $t+1 = u$

$t = u-1$

$dt = du$

* Pembilang $\longrightarrow t^2+1 = (u-1)^2+1$

$= u^2-2u+1+1$

$= u^2-2u+2$

▶ Substitusi ke Integral

$$\int \frac{u^2-2u+2}{(u^2+2)^2}\,du = \int \frac{u^2}{(u^2+2)^2}\,du - \int \frac{2u}{(u^2+2)^2}\,du + \int \frac{2}{(u^2+2)^2}\,du$$

* Penyelesaian $\int \frac{u^2}{(u^2+2)^2}\,du$

$$\frac{u^2}{(u^2+2)^2} = \frac{A}{u^2+2} + \frac{B}{(u^2+2)^2}$$

$$u^2 = A(u^2+2) + B$$

$A = 1 \longrightarrow 2A + B = 0$

$2 + B = 0$

$B = -2$

$$\int \frac{u^2}{(u^2+2)^2}\,du = \int \frac{1}{u^2+2}\,du - 2\int \frac{du}{(u^2+2)^2}$$

▶ Substitusi ke Integral

↳ $$\int \frac{u^2}{(u^2+2)^2}\,du - 2\int \frac{u}{(u^2+2)^2}\,du + \int \frac{2}{(u^2+2)^2}\,du$$

$$= \int \frac{du}{u^2+2}\,du - \underbrace{2\int \frac{du}{(u^2+2)^2}} - 2\int \frac{u}{(u^2+2)^2}\,du + \underbrace{2\int \frac{du}{(u^2+2)^2}}$$

karena saling menghilangkan

$$= \int \frac{1}{u^2+2}\,du - 2\int \frac{u}{(u^2+2)^2}\,du$$

* Penyelesaian $2\int \frac{u}{(u^2+2)^2}\,du$

misal : $u^2+2 = t$

$2u\,du = dt$

ud

$\rightarrow \int \frac{dt}{t^2}$

$= -\frac{1}{t} + C$

$= -\frac{1}{(u^2+2)} + C$

Hasil Akhir

$\rightarrow = \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} - \frac{1}{(u^2+2)} + C$

↳ $\frac{1}{\sqrt{2}}\tan^{-1}\frac{(t+1)}{\sqrt{2}} - \frac{1}{(t+1)^2+2} + C$

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Bab 2.3 Teknik Integrasi yang Lain

▶ Pemisalan

$\sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$\sqrt{a^2+x^2} \longrightarrow x = a\tan\theta$

$\sqrt{x^2-a^2} \longrightarrow x = a\sec\theta$

※ Konsep! : Tujuan utamanya untuk menghilangkan bentuk akar yang rumit.

misal : $\sqrt{1-x^2}$, dengan pemisalan $x = \sin\theta$, maka

$\sqrt{1-\sin^2\theta} \longrightarrow \sqrt{\cos^2\theta}$
$= \cos\theta.$

Sehingga bentuknya dapat diselesaikan mudah.

- $\sqrt{a^2-x^2}$ gunakan pemisalan $x = a\sin\theta$ karena menghasilkan $1-\sin^2\theta \simeq \cos^2\theta$
- $\sqrt{a^2+x^2} \longrightarrow x = a\tan\theta$ karena
  $\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow \tan^2\theta + 1 = \sec^2\theta$
  $\sqrt{\tan^2\theta+1}$, sehingga menghasilkan $\sec^2\theta$
- $\sqrt{x^2-a^2}$ Gunakan pemisalan $x = a\sec\theta$, karena $\sec^2\theta - 1 = \tan^2\theta$.

▶ Contoh Soal

1). a. $\int\sqrt{1-x^2}\,dx$ misal : $x = \sin\theta$
$dx = \cos\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \int\sqrt{1-\sin^2\theta}\,.\cos\theta\, d\theta$

$= \int\sqrt{\cos^2\theta}\,\cos\theta\, d\theta$

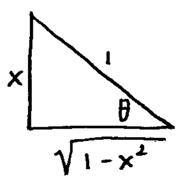
$= \int\cos^2\theta\, d\theta$

$= \int\frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \frac{1}{2}\int 1+\cos 2\theta\, d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

$\theta = \sin^{-1} x$



$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\,.\,x\,.\sqrt{1-x^2}$

$= \frac{1}{2}\left[\sin^{-1}(x) + \cancel{\frac{1}{2}}\cdot\cancel{2}\,.\,x\sqrt{1-x^2}\right] + C$

$= \frac{1}{2}\left[\sin^{-1}x + x\sqrt{1-x^2}\right] + C$ //

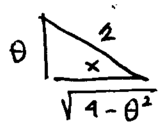
g). $\int\frac{\theta^2}{\sqrt{4-\theta^2}}\,d\theta$

$\cos 2x = 1 - 2\sin^2 x$
$\sin^2 x = \frac{1-\cos 2x}{2}$

misal : $\theta = 2\sin x$
$d\theta = 2\cos x\, dx$

$X = \sin^{-1}\frac{\theta}{2}$

$= \int\frac{4\sin^2 x}{\sqrt{4(1-\sin^2 x)}}\,.\,2\cos x\, dx$



$= \int\frac{4\sin^2 x}{\cancel{2\cos x}}\,.\,\cancel{2\cos x}\, dx$

$2\sin x\cos x = \cancel{2}.\frac{\theta}{\cancel{2}}.\frac{\sqrt{4-\theta^2}}{2}$
$= \frac{\theta\sqrt{4-\theta^2}}{2}$

$= 4\int\frac{1}{2}(1-\cos 2x)\, dx$

$= 2\int 1 - \cos 2x\, dx$

$= 2.\left[x - \frac{1}{2}\sin 2x\right] + C \rightarrow$ //

$\hookrightarrow 2\left[\sin^{-1}\left(\frac{\theta}{2}\right) - \frac{\theta\sqrt{4-\theta^2}}{2}\right] + C$

h). $\int\frac{1}{t^2\sqrt{t^2-9}}\,dt$

karena terdapat bentuk $t^2-9 \simeq u^2-a^2$, maka misalkan

$t = 3\sec\theta \longrightarrow dt = 3.\sec\theta\tan\theta\, d\theta$

$= \int\frac{1}{9\sec^2\theta\sqrt{9(\sec^2\theta-1)}}\,.\,3\sec\theta\tan\theta\, d\theta$

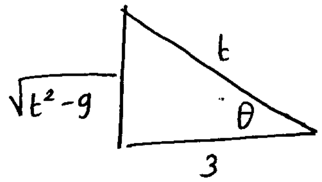
$= \int\frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9.\tan^2\theta}}\, d\theta$

$\sec\theta = \frac{t}{3}$

$\theta = \sec^{-1}\left(\frac{t}{3}\right)$

$= \int\frac{\cancel{3\sec\theta\tan\theta}}{9.\cancel{3}.\sec^{\cancel{2}}\theta.\cancel{\tan\theta}}\, d\theta$

$= \frac{1}{9}\int\frac{d\theta}{\sec\theta}$



$= \frac{1}{9}\int\cos\theta\, d\theta$

$= \frac{1}{9}.\sin\theta + C$

$= \frac{1}{9}\,\frac{\sqrt{t^2-9}}{t} + C$ //

m). $\displaystyle\int \frac{1}{x^2\sqrt{9x^2-4}}\,dx$

misal : $9x^2 = u^2$
$3x = u$
$3\,dx = du$
$dx = \dfrac{du}{3}$

karena terdapat $u^2-4 \approx u^2-a^2$, maka diperoleh pemisalan.
misal $u = 2\sec\theta$
$du = 2.\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{\left(\frac{u}{3}\right)^2\sqrt{u^2-4}}\,du$$

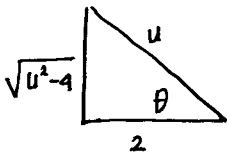
$$= \int \frac{1}{\frac{u^2}{9}\sqrt{u^2-4}}\,du$$

$$= \int \frac{1}{\frac{\cancel{4}\sec^2\theta}{9}\sqrt{4(\sec^2\theta-1)}}\,\cancel{d\theta}\,.\,\cancel{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{9}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\,2\sqrt{\tan^2\theta}} \qquad \theta = \sec^{-1}\left(\frac{u}{2}\right)$$

$$= \frac{9}{4}\int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}}{\cancel{\sec^2\theta}\,.\,\cancel{\tan\theta}}\,d\theta$$

$$= \frac{9}{4}\int \frac{1}{\sec\theta}\,d\theta$$



$$= \frac{9}{4}\int \cos\theta\,d\theta$$

$$= \frac{9}{4}.\sin\theta + C$$

$$= \frac{9}{4}.\frac{\sqrt{u^2-4}}{u} + C$$

$$= \frac{\cancel{9}^{\,3}}{4}\,\frac{\sqrt{9x^2-4}}{\cancel{3}x} + C$$

$$= \frac{3}{4}\sqrt{9x^2-4} + C$$

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p). $\displaystyle\int \frac{1}{t^2\sqrt{t^2+9}}\,dt$

karena bentuk $t^2+9 \approx a^2+x^2$

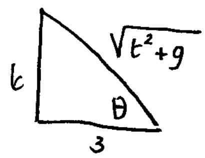
misal : $t = 3\tan\theta \;\longrightarrow\; \tan^{-1}\left(\frac{t}{3}\right) = \theta$
$dt = 3\sec^2\theta\,d\theta$

$$\int \frac{\cancel{3}\sec^2\theta}{3\,\cancel{9}\tan^2\theta\sqrt{9(\tan^2\theta+1)}}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cancel{\sec^2\theta}^{\,\sec\theta}}{\tan^2\theta\,\cancel{\sec\theta}}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos^2\theta}}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$



misal : $u = \sin\theta$
$du = \cos\theta\,d\theta$

$$= \frac{1}{9}\int \frac{du}{u^2}$$

$$= \frac{1}{9}\left[-\frac{1}{u}\right] + C$$

$$= -\frac{1}{9}\cdot\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{9}\cdot\frac{\sqrt{t^2+9}}{t} + C$$

$$= -\frac{\sqrt{t^2+9}}{9t} + C$$

Latihan Soal!

1. b. $\displaystyle\int \sqrt{1-9t^2}\,dt$
   d. $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$
   e. $\displaystyle\int \frac{4}{(1+x^2)^2}\,dx$
   l. $\displaystyle\int \frac{2\theta^2}{\sqrt{1+\theta^2}}\,d\theta$
   q. $\displaystyle\int \frac{1}{\sqrt{t^2-25}}\,dt$
   v. $\displaystyle\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}}\,d\theta$

5. d. $\displaystyle\int \frac{1}{6x^2+6x+5}\,dx$
   e. $\displaystyle\int \frac{1}{\sqrt{x^2-4x+10}}\,dx$

Integral Mencakup ax^2+bx+C

Langkah-Langkah :

1. Buat Kuadrat Sempurna

   ex : $(x-2)^2+\dots$

   $(x+3)^2+\dots$

2. Lakukan substitusi yang sesuai

Latihan Soal!

5). a. $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$

- Buat kuadrat sempurna

  $x^2-4x+13$

  kuadrat sempurna dari (x^2-4x) dapat dibentuk dengan menambahkan 4 di belakang.

  * Ide : bagi b dengan 2, karena hasil -2, maka

  $(x-2)^2 = x^2-4x+4$

Jadi, bentuk pada soal dapat diubah menjadi

$$\int \frac{1}{(x-2)^2+9}\,dx = \int \frac{1}{(x-2)^2+3^2}\,dx$$

misal : $x-2=p$

$dx = dp$

$$= \int \frac{dp}{p^2+3^2}$$

karena bentuknya sesuai dengan $\int \frac{dp}{p^2+a^2}$, maka hasilnya $\frac{1}{a}\tan^{-1}\frac{p}{a}$.

$$= \frac{1}{3}\tan^{-1}\frac{p}{3}+C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$$

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b. $\displaystyle\int \frac{1}{\sqrt{2t-t^2}}\,dt$

$$-(t^2-2t) = -(t-1)(t-1)+1$$
$$= -(t^2-2t+1)+1$$
$$= 1-(t-1)^2$$

$$\int \frac{1}{\sqrt{1-(t-1)^2}}\,dt$$

misal : $u = t-1$

$du = dt$

$$\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}(u)+C$$

$$= \sin^{-1}(t-1)+C$$

//

f. $\displaystyle\int \frac{x}{x^2+6x+3}\,dx$

$x^2+6x+3 \rightarrow (x+3)^2-6$

misal : $t = x+3 \rightarrow x = t-3$

$dt = dx$

$$\int \frac{x}{(x+3)^2-6}\,dx$$

$$= \int \frac{t-3}{t^2-6}\,dt$$

$$= \int \frac{t}{t^2-6}\,dt - \int \frac{3}{t^2-6}\,dt$$

* $\displaystyle\int \frac{t}{t^2-6}\,dt$

misal : $u = t^2-6$

$du = 2t\,dt$

$t\,dt = \frac{du}{2}$

$$\int \frac{du/2}{u} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|+C$$

$$= \frac{1}{2}\ln|t^2-6|+C$$

* $\displaystyle\int \frac{3}{t^2-6}\,dt$

$t^2-6 = (t+\sqrt{6})(t-\sqrt{6})$

$$\frac{3}{t^2-6} = \frac{A}{t+\sqrt{6}} + \frac{B}{t-\sqrt{6}}$$

$$3 = A(t-\sqrt{6}) + B(t+\sqrt{6})$$

$$3 = At + Bt - A\sqrt{6} + B\sqrt{6}$$

- $A+B = 0$
- $(B-A)\sqrt{6} = 3$

  $B-A = \frac{1}{2}\sqrt{6}$

$$\begin{array}{r} A+B = 0 \\ -A+B = \frac{1}{2}\sqrt{6} \\ \hline 2B = \frac{1}{2}\sqrt{6} \end{array} \;+ \quad \rightarrow B = \frac{\sqrt{6}}{4}$$

$$A = -\frac{\sqrt{6}}{4}$$

$$\int \frac{3}{t^2-6}\,dt = -\int \frac{\sqrt{6}/4}{t+\sqrt{6}}\,dt + \int \frac{\sqrt{6}/4}{t-\sqrt{6}}\,dt$$

$$= -\frac{\sqrt{6}}{4}\cdot \ln|t+\sqrt{6}| + \frac{\sqrt{6}}{4}\ln|t-\sqrt{6}| + C$$

$$= \frac{\sqrt{6}}{4}\cdot \ln \frac{|t-\sqrt{6}|}{|t+\sqrt{6}|} + C$$

Sehingga, diperoleh :

$$\int \frac{x}{x^2+6x+3}\,dx = \frac{1}{2}\ln|t^2-6| + \frac{\sqrt{6}}{4}\ln\frac{|t-\sqrt{6}|}{|t+\sqrt{6}|} + C$$

//

Integral yang Memuat Pangkat Rasional

Ide : menyederhanakan lewat substitusi

$$u = x^{1/n}$$

* n KPK dari pengebut dalam pangkat x.

<u>Contoh soal</u> :

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{x^{1/2}}{1+x^{1/3}}\,dx$$

Amati pangkat $\longrightarrow x^{1/2}, x^{1/3}$. Ambil KPK dari 2 dan 3, yaitu 6.

Sehingga, diperoleh :

misal : $u = x^{1/6}$

$x = u^6$

$dx = 6u^5\,du$

$$\int \frac{x^{1/2}}{1+x^{1/3}}\,dx = \int \frac{u^{6/2}}{1+u^{6/3}} \cdot 6u^5\,du$$

$$= 6 \int \frac{u^3 . u^5}{1+u^2}\,du$$

$$= 6\int \frac{u^8}{1+u^2}\,du$$

karena derajat (pangkat tertinggi) pembilang lebih besar dari derajat (pangkat tertinggi) penyebut, maka lakukan pembagian polinomial terlebih dahulu

$$\begin{array}{r|l} & u^6 - u^4 + u^2 - 1 \\ 1+u^2 & u^8 \\ & u^8 + u^6 \quad - \\ & -u^6 \\ & -u^6 - u^4 \quad - \\ & u^4 \\ & u^4 + u^2 \quad - \\ & -u^2 \\ & -u^2 - 1 \quad - \\ & 1 \end{array}$$

$$6\int \frac{u^8}{1+u^2}\,du = 6\left[\int u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\,du\right]$$

$$= 6\left[\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1}u\right] + C$$

$$= 6\left[\frac{1}{7}x^{7/6} - \frac{1}{5}x^{5/6} + \frac{1}{3}x^{1/2} - x^{1/6} + \tan^{-1}\left(x^{1/6}\right)\right] + C$$

//

5). x). $\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

misal : $u = x^{1/6}$

$x = u^6$

$dx = 6u^5\,du$

$$\int \frac{6u^5}{u^3-u^2}\,du = 6\int \frac{u^5}{u^3-u^2}\,du$$

$$= 6\int \frac{u^{\cancel{5}\,3}}{\cancel{u^2}(u-1)}\,du$$

$$= 6\int \frac{u^3}{u-1}\,du$$

$$\begin{array}{r|l} & u^2 + u + 1 \\ u-1 & u^3 \\ & u^3 - u^2 \quad - \\ & u^2 \\ & u^2 - u \quad - \\ & u \\ & u - 1 \quad - \\ & 1 \end{array}$$

$$= 6\int u^2 + u + 1 + \frac{1}{u-1}\,du$$

$$= 6.\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$$

$$= 6\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$$

$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6} - 1| + C$$

//

Latihan Soal : 5. y). $\int \frac{1}{t + t^{3/5}}\,dt$

aa). $\int \frac{t^{2/3}}{t+1}\,dt$

Bab 2.3 Teknik Integrasi yang Lain

▶ Pemisalan

$\sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$\sqrt{a^2+x^2} \longrightarrow x = a\tan\theta$

$\sqrt{x^2-a^2} \longrightarrow x = a\sec\theta$

* Konsep! : Tujuan utamanya untuk menghilangkan bentuk akar yang rumit.

misal : $\sqrt{1-x^2}$, dengan pemisalan $x = \sin\theta$, maka

$\sqrt{1-\sin^2\theta} \longrightarrow \sqrt{\cos^2\theta}$
$= \cos\theta$.

Sehingga bentuknya dapat diselesaikan mudah.

- $\sqrt{a^2-x^2}$ gunakan pemisalan $x = a\sin\theta$ karena menghasilkan $1-\sin^2\theta \approx \cos^2\theta$
- $\sqrt{a^2+x^2} \longrightarrow x = a\tan\theta$ karena
  $\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \longrightarrow \tan^2\theta + 1 = \sec^2\theta$
  $\sqrt{\tan^2\theta+1}$, sehingga menghasilkan $\sec^2\theta$
- $\sqrt{x^2-a^2}$ Gunakan pemisalan $x = a\sec\theta$, karena
  $\sec^2\theta - 1 = \tan^2\theta$.

▶ Contoh Soal

1). a. $\int\sqrt{1-x^2}\,dx$ misal : $x = \sin\theta$
$dx = \cos\theta\,d\theta$

$= \int\sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$ $\cos 2\theta = 2\cos^2\theta - 1$

$= \int\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$= \int\cos^2\theta\,d\theta$

$= \int\frac{1}{2}(1+\cos 2\theta)\,d\theta$

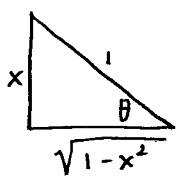
$= \frac{1}{2}\int 1+\cos 2\theta\,d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

$= \frac{1}{2}\left[\sin^{-1}(x) + \frac{1}{\cancel{2}}.\cancel{2}.x\sqrt{1-x^2}\right] + C$

$= \frac{1}{2}\left[\sin^{-1}x + x\sqrt{1-x^2}\right] + C$ //

$\theta = \sin^{-1}x$



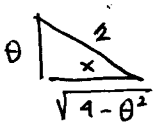
$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2.x.\sqrt{1-x^2}$

g). $\int\frac{\theta^2}{\sqrt{4-\theta^2}}\,d\theta$ $\cos 2x = 1-2\sin^2 x$
$\sin^2 x = \frac{1-\cos 2x}{2}$

misal : $\theta = 2\sin x$

$d\theta = 2\cos x\,dx$ $x = \sin^{-1}\frac{\theta}{2}$

$= \int\frac{4\sin^2 x}{\sqrt{4(1-\sin^2 x)}}\,.\,2\cos x\,dx$



$= \int\frac{4\sin^2 x}{\cancel{2\cos x}}\,.\,\cancel{2\cos x}\,dx$

$2\sin x\cos x = \cancel{2}.\frac{\theta}{\cancel{2}}.\frac{\sqrt{4-\theta^2}}{2}$
$= \frac{\theta\sqrt{4-\theta^2}}{2}$

$= 4\int\frac{1}{2}(1-\cos 2x)\,dx$

$= 2\int 1-\cos 2x\,dx$

$= 2.\left[x - \frac{1}{2}\sin 2x\right] + C \longrightarrow$ //

↳ $2\left[\sin^{-1}\left(\frac{\theta}{2}\right) - \frac{\theta\sqrt{4-\theta^2}}{2}\right] + C$

h). $\int\frac{1}{t^2\sqrt{t^2-9}}\,dt$

karena terdapat bentuk $t^2-9 \approx u^2-a^2$, maka misalkan

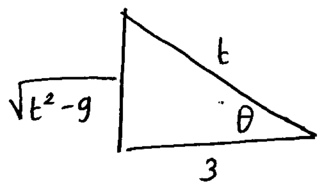
$t = 3\sec\theta \longrightarrow dt = 3.\sec\theta\tan\theta\,d\theta$

$= \int\frac{1}{9\sec^2\theta\sqrt{9(\sec^2\theta-1)}}\,.\,3\sec\theta\tan\theta\,d\theta$

$= \int\frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9.\tan^2\theta}}\,d\theta$ $\sec\theta = \frac{t}{3}$

$= \int\frac{\cancel{3\sec\theta\tan\theta}}{9.\cancel{3}.\cancel{\sec^2\theta}.\cancel{\tan\theta}}\,d\theta$ $\theta = \sec^{-1}\left(\frac{t}{3}\right)$

$= \frac{1}{9}\int\frac{d\theta}{\sec\theta}$



$= \frac{1}{9}\int\cos\theta\,d\theta$

$= \frac{1}{9}.\sin\theta + C$

$= \frac{1}{9}\,\frac{\sqrt{t^2-9}}{t} + C$ //

m). $\int \frac{1}{x^2\sqrt{9x^2-4}}\,dx$

misal : $9x^2 = u^2$

$3x = u$

$3dx = du$

$dx = \frac{du}{3}$

karena terdapat $u^2-4 \approx u^2-a^2$, maka diperoleh pemisalan.

misal $u = 2\sec\theta$

$du = 2.\sec\theta\tan\theta\, d\theta$

$$\int \frac{1}{\left(\frac{u}{3}\right)^2\sqrt{u^2-4}}\,du$$

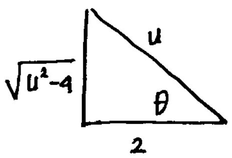
$$= \int \frac{1}{\frac{u^2}{9}\sqrt{u^2-4}}\,du$$

$$= \int \frac{1}{\frac{\cancel{2}4\sec^2\theta}{9}\sqrt{4(\sec^2\theta-1)}}\,\cancel{du}\,.\cancel{2}\sec\theta\tan\theta\, d\theta$$

$$= \frac{9}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\, 2\sqrt{\tan^2\theta}} \qquad \theta = \sec^{-1}\left(\frac{u}{2}\right)$$

$$= \frac{9}{4}\int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}}{\cancel{\sec^2\theta}\,.\,\cancel{\tan\theta}}\,d\theta$$

$$= \frac{9}{4}\int \frac{1}{\sec\theta}\,d\theta$$



$$= \frac{9}{4}\int \cos\theta\, d\theta$$

$$= \frac{9}{4}.\sin\theta + C$$

$$= \frac{9}{4}\cdot\frac{\sqrt{u^2-4}}{u} + C$$

$$= \frac{\cancel{9}\,3}{4}\,\frac{\sqrt{9x^2-4}}{\cancel{3}x} + C$$

$$= \frac{3}{4}\sqrt{9x^2-4} + C$$

p). $\int \frac{1}{t^2\sqrt{t^2+9}}\,dt$

karena bentuk $t^2+9 \approx a^2+x^2$

misal : $t = 3\tan\theta \quad \rightarrow \quad \tan^{-1}\left(\frac{t}{3}\right) = \theta$

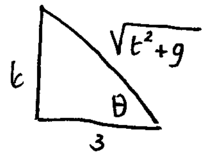
$dt = 3\sec^2\theta\, d\theta$

$$\int \frac{\cancel{3}\sec^2\theta}{3\,\cancel{9}\tan^2\theta\sqrt{9(\tan^2\theta+1)}}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cancel{\sec^2\theta}^{\,\sec\theta}}{\tan^2\theta\,\cancel{\sec\theta}}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos^2\theta}}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$



misal : $u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{9}\int \frac{du}{u^2}$$

$$= \frac{1}{9}\left[-\frac{1}{u}\right] + C$$

$$= -\frac{1}{9}\cdot\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{9}\cdot\frac{\sqrt{t^2+9}}{t} + C$$

$$= -\frac{\sqrt{t^2+9}}{9t} + C$$

Latihan Soal!

1. b. $\int\sqrt{1-9t^2}\,dt$

   d. $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$

   e. $\int \frac{4}{(1+x^2)^2}\,dx$

   l. $\int \frac{2\theta^2}{\sqrt{1+\theta^2}}\,d\theta$

   q. $\int \frac{1}{\sqrt{t^2-25}}\,dt$

   v. $\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}}\,d\theta$

5. d. $\int \frac{1}{6x^2+6x+5}\,dx$

   e. $\int \frac{1}{\sqrt{x^2-4x+10}}\,dx$

Integral Mencakup ax^2+bx+C

Langkah-Langkah :

1. Buat Kuadrat Sempurna

   ex : $(x-2)^2+\dots$

   $(x+3)^2+\dots$

2. Lakukan Substitusi yang sesuai

Latihan Soal!

5). a. $\int \frac{1}{x^2-4x+13}\,dx$

- Buat kuadrat sempurna

$x^2-4x+13$

kuadrat sempurna dari (x^2-4x) dapat dibentuk dengan menambahkan 4 di belakang.

* Ide : bagi b dengan 2, karena hasil -2, maka

$(x-2)^2 = x^2-4x+4$

Jadi, bentuk pada soal dapat diubah menjadi

$$\int \frac{1}{(x-2)^2+9}\,dx = \int \frac{1}{(x-2)^2+3^2}\,dx$$

misal : $x-2 = p$

$dx = dp$

$$= \int \frac{dp}{p^2+3^2}$$

karena bentuknya sesuai dengan $\int \frac{dp}{p^2+a^2}$, maka

hasilnya $\frac{1}{a}\tan^{-1}\frac{p}{a}$.

$$= \frac{1}{3}\tan^{-1}\frac{p}{3} + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

//

b. $\int \frac{1}{\sqrt{2t-t^2}}\,dt$

$$-(t^2-2t) = -(t-1)(t-1)+1$$
$$= -(t^2-2t+1)+1$$
$$= 1-(t-1)^2$$

$$\int \frac{1}{\sqrt{1-(t-1)^2}}\,dt$$

misal : $u = t-1$

$du = dt$

$$\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}(u) + C$$

$$= \sin^{-1}(t-1) + C$$

//

f. $\int \frac{x}{x^2+6x+3}\,dx$

$x^2+6x+3 \rightarrow (x+3)^2-6$

misal : $t = x+3 \rightarrow x = t-3$

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$$\int \frac{x}{(x+3)^2-6}\,dx$$

$$= \int \frac{t-3}{t^2-6}\,dt$$

$$= \int \frac{t}{t^2-6}\,dt - \int \frac{3}{t^2-6}\,dt$$

* $\int \frac{t}{t^2-6}\,dt$

misal : $u = t^2-6$

$du = 2t\,dt$

$t\,dt = \frac{du}{2}$

$$\int \frac{du/2}{u} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|t^2-6| + C$$

* $\int \frac{3}{t^2-6}\,dt$

$t^2-6 = (t+\sqrt{6})(t-\sqrt{6})$

$$\frac{3}{t^2-6} = \frac{A}{t+\sqrt{6}} + \frac{B}{t-\sqrt{6}}$$

$$3 = A(t-\sqrt{6}) + B(t+\sqrt{6})$$

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$A = -\frac{\sqrt{6}}{4}$

$$\int \frac{3}{t^2-6}\,dt = -\int \frac{\sqrt{6}/4}{t+\sqrt{6}}\,dt + \int \frac{\sqrt{6}/4}{t-\sqrt{6}}\,dt$$

$$= -\frac{\sqrt{6}}{4}\cdot \ln\left|t+\sqrt{6}\right| + \frac{\sqrt{6}}{4}\ln\left|t-\sqrt{6}\right| + C$$

$$= \frac{\sqrt{6}}{4}\cdot \ln \frac{|t-\sqrt{6}|}{|t+\sqrt{6}|} + C$$

Sehingga, diperoleh :

$$\int \frac{x}{x^2+6x+3}\,dx = \frac{1}{2}\ln\left|t^2-6\right| + \frac{\sqrt{6}}{4}\ln\frac{|t-\sqrt{6}|}{|t+\sqrt{6}|} + C$$

//

▶ Integral yang Memuat Pangkat Rasional

Ide : menyederhanakan lewat substitusi

$$u = x^{1/n}$$

* n KPK dari penyebut dalam pangkat x.

Contoh soal :

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{x^{1/2}}{1+x^{1/3}}\,dx$$

Amati pangkat $\longrightarrow x^{1/2}, x^{1/3}$. Ambil KPK dari 2 dan 3, yaitu 6.

Sehingga, diperoleh :

misal : $u = x^{1/6}$

$x = u^6$

$dx = 6u^5\,du$

$$\int \frac{x^{1/2}}{1+x^{1/3}}\,dx = \int \frac{u^{6/2}}{1+u^{6/3}}\cdot 6u^5\,du$$

$$= 6\cdot\int \frac{u^3\cdot u^5}{1+u^2}\,du$$

$$= 6\int \frac{u^8}{1+u^2}\,du$$

karena derajat (pangkat tertinggi) pembilang lebih besar dari derajat (pangkat tertinggi) penyebut, maka lakukan pembagian polinomial terlebih dahulu

$$\begin{array}{l} \qquad u^6 - u^4 + u^2 \\ 1+u^2 \,\overline{)\, u^8} \\ \qquad u^8 + u^6 \quad - \\ \hline \qquad -u^6 \\ \qquad -u^6 - u^4 \quad - \\ \hline \qquad u^4 \end{array}$$

$$\begin{array}{l} \qquad u^6 - u^4 + u^2 - 1 \\ 1+u^2 \,\overline{)\, u^8} \\ \qquad u^8 + u^6 \quad - \\ \hline \qquad -u^6 \\ \qquad -u^6 - u^4 \quad - \\ \hline \qquad u^4 \\ \qquad u^4 + u^2 \quad - \\ \hline \qquad -u^2 \\ \qquad -u^2 - 1 \quad - \\ \hline \qquad 1 \end{array}$$

$$\bullet\ 6\int \frac{u^8}{1+u^2}\,du = 6\left[\int u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\,du\right]$$

$$= 6\left[\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1}u\right] + C$$

$$= 6\left[\frac{1}{7}x^{7/6} - \frac{1}{5}x^{5/6} + \frac{1}{3}x^{1/2} - x^{1/6} + \tan^{-1}\left(x^{1/6}\right)\right] + C$$

//

5). x). $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

misal : $u = x^{1/6}$

$x = u^6$

$dx = 6u^5\,du$

$$\int \frac{6u^5}{u^3-u^2}\cdot du = 6\int \frac{u^5}{u^3-u^2}\,du$$

$$= 6\int \frac{u^{\cancel{5}\,3}}{\cancel{u^2}(u-1)}\,du$$

$$= 6\int \frac{u^3}{u-1}\,du$$

$$\begin{array}{l} \qquad u^2 + u + 1 \\ u-1 \,\overline{)\, u^3} \\ \qquad u^3 - u^2 \quad - \\ \hline \qquad u^2 \\ \qquad u^2 - u \quad - \\ \hline \qquad u \\ \qquad u - 1 \quad - \\ \hline \qquad 1 \end{array}$$

$$= 6\int u^2 + u + 1 + \frac{1}{u-1}\,du$$

$$= 6\cdot\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$$

$$= 6\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$$

$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6} - 1\right| + C$$

//

Latihan Soal : 5. y). $\displaystyle\int \frac{1}{t + t^{3/5}}\,dt$

aa). $\displaystyle\int \frac{t^{2/3}}{t+1}\,dt$